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Analytical Formulations for Coupled Axial-Bending-Shear-Torsion Vibrations of Asymmetric Laminated Timoshenko Beams

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Abstract

This paper presents the exact closed-form solutions for the vibration analysis of extension-bending-shear-torsion coupled responses in asymmetric laminated Timoshenko beams. The study derives four governing differential equations and corresponding boundary conditions using Hamilton's variational principle. The formulation, based on first-order shear deformation theory (*FSDT*), accounts for rotary inertia, Poisson's ratio, and extensional-flexural-torsional-shear coupling effects arising from material anisotropy. The exact solutions are obtained for various harmonic bending and twisting excitations under different boundary conditions, capturing the fully coupled axial, bending, and torsional responses. This approach offers a robust analytical method for understanding the complex dynamic behavior of laminated composite beams subjected to dynamic loads, with potential applications in aerospace, mechanical, and civil engineering structures, such as aircraft components, turbine blades, and bridges.

Keywords: Closed-form solution; extension-bending-shear-torsion coupled response; harmonic excitations; asymmetric laminated beam.

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الصيغ التحليلية للاهتزازات المحورية والانحناء والقص والالتواء المقترنة لعوارض تيموشينكو ذات الصفائح المركبة الغير متماثلة

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الملخص

تقدم هذه الورقة حلولاً دقيقة ذات شكل مغلق لتحليل الاهتزاز للاستجابات المقترنة بالانحناء والقص والالتواء في عارضات تيموشينكو ذات الرقائق المركبة غير المتماثلة. تستمد الدراسة أربع معادلات تفاضلية حاكمة وشروط حدودية مقابلة باستخدام مبدأ هاملتون التبايني. تعتمد الصيغة على نظرية تشوه القص من الدرجة الأولى (FSDT)، وتأخذ في الاعتبار القصور الذاتي الدوار، ونسبة بواسون، وتأثيرات اقتران القص الممتدة والانحناء والالتواء الناشئة عن تباين المواد. يتم الحصول على الحلول الدقيقة لمختلف إثارات الانحناء والالتواء التوافقي في ظل ظروف حدودية مختلفة، مما يلتقط الاستجابات المحورية والانحناء والالتوائية المقترنة بالكامل. يقدم هذا النهج طريقة تحليلية قوية لفهم السلوك الديناميكي المعقد للحزم المركبة المصفحة المعرضة للأحمال الديناميكية، مع التطبيقات المحتملة في هياكل الهندسة الفضائية والميكانيكية والمدنية، مثل مكونات الطائرات وشفرات التوربينات والجسور.

الكلمات المفتاحية: حل مغلق الشكل؛ استجابة مقترنة بالامتداد والانحناء والقص والالتواء؛ الإثارات التوافقية؛ عارضات الرقائق المركبة الغير متماثلة.

Introduction and Objective

Composite laminate structural members are increasingly used in various engineering applications due to their excellent strength-to-weight and stiffness-to-weight ratios. Multilayered composite beams are widely applied in aerospace, mechanical, and civil engineering, including in aircraft wings, fuselage structures, helicopter blades, vehicle axles, turbine blades, and marine structural frames, due to their superior properties. In these applications, composite laminated beams are often subjected to cyclic dynamic loading, such as harmonic excitations, which can arise from aerodynamic forces, hydrodynamic wave motions, wind loads, or imbalances in rotating and reciprocating machinery. Harmonic forces can induce unwanted vibrations and lead to fatigue failure, which has become an important consideration in the design of composite laminated structures. Under such forces, the transient component of the dynamic response is most significant at the start but quickly dissipates, making it less relevant for fatigue life evaluation. On the other hand, the steady-state dynamic response persists over time and is important for fatigue analysis, which is the primary focus of this study. The objective is to develop an exact, efficient solution that isolates and captures the steady-state response. This analytical closed-form solution also predicts the quasi-static responses, eigen-frequencies, and eigen-modes of asymmetric laminated composite beams.

While dynamic analysis of composite asymmetric laminated beams has received attention in recent research, most studies have focused on free vibrations rather than responses to dynamic forces. Various studies have developed exact analytical solutions and finite element techniques for free vibration analysis of composite beams. For instance, Khdeir and Reddy (1980) proposed an exact solution based on higher-order shear deformation theory for cross-ply rectangular beams, while Banerjee and Williams (1996) used the exact dynamic stiffness matrix to study composite laminated beam vibrations, accounting for shear deformation and bending-torsional coupling. In 1998, Banerjee further explored the free vibration of axially composite laminated Timoshenko beams using the dynamic stiffness matrix method, where his formulation captured the coupling between bending and torsion, as well as the effects of axial force, shear deformation, and rotary inertia. Chakraborty et al.

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(2002) employed finite element analysis based on first-order shear deformation theory to study free vibration and wave propagation in composite laminated beams with symmetric and asymmetric ply stacking. Murthy et al. (2005) developed a refined two-node beam element based on higher-order shear deformation theory for the coupled axial-flexural-shear vibration analysis of asymmetrically stacked composite beams. Their formulation ensured that the finite element shape functions satisfied static equilibrium governing equations. Tahani (2007) proposed a displacement-based layerwise beam theory and applied it to cross-ply antisymmetric (0°/90°) and (0°/90°/0°) laminated beams under sinusoidal loads. Jun et al. (2008) developed the exact dynamic stiffness matrix method to analyze the free vibration of arbitrary laminated composite beams, utilizing first-order shear deformation, trigonometric shear deformation, and higher-order shear deformation beam theories. Their mathematical formulations considered the effects of shear deformation, rotary inertia, Poisson's ratio, and extensional-bending coupling deformations. Hjaji et al. (2016) developed a super-convergent one-dimensional finite beam element with two nodes to analyze the steady-state dynamic flexural response of symmetric laminated composite beams under bending harmonic forces. This new beam element, based on exact shape functions that precisely satisfy the dynamic coupled governing equations, is applicable to symmetric laminated composite beams and takes into account shear deformation, rotary inertia, and Poisson’s ratio. In 2017, Hjaji et al. extended their work by investigating analytical closed-form solutions for the flexural dynamic analysis of symmetric laminated composite beams subjected to transverse harmonic forces. Their formulations, based on first-order shear deformation theory, incorporate the effects of shear deformation, rotary inertia, Poisson’s ratio, and fiber orientation. Nguyen et al. (2017) presents a new analytical solution utilizing a higher-order beam theory for the static, buckling, and vibration analysis of laminated composite beams. The governing equations of motion are derived from Lagrange's equations. An analytical solution employing trigonometric series, which accommodates various boundary

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conditions, is formulated to address the problem. In 2018, Nguyen et al. extended their work to investigate Ritz-based solutions for analyzing the bending, buckling, and vibration behaviors of laminated composite beams with arbitrary lay-ups. A quasi-3D theory is employed, which considers a higher-order variation of both axial and transverse displacements to effectively capture the influences of shear and normal deformations on the behavior of composite beams. Karkon (2020) developed a new three-node element with three degrees of freedom per node for the bending, free vibration, and buckling analysis of laminated beams. The formulation of the element is based on the first-order shear deformation theory (FSDT). Akbas et al. (2021) performed a dynamic analysis of fiber-reinforced composite laminated simply-supported beams under moving loads, using Timoshenko Beam Theory in combination with the Newmark average acceleration and Ritz methods. Recently, Hjaji and Nagiar (2023) investigated the dynamic analysis for extensional-bending coupled vibration responses of antisymmetric composite laminated rectangular beams under various bending forces. Their formulation was based on the first shear deformable beam theory which accounts for the effects of rotary inertia, Poisson's ratio, fiber orientation and the bending-extension coupling coming from the material anisotropy. More recently, Hjaji and Nagiar (2024) developed an exact one-dimensional finite beam element with three degrees of freedom per node for analyzing extensional-bending coupled vibrations in antisymmetric composite laminated beams subjected to various harmonic axial and bending forces. The new beam element based on the exact shape functions which exactly satisfy the dynamic coupled governing filed equations were derived in previous study of Hjaji and Nagiar (2023) is applicable to antisymmetric laminated composite beams and accounts for the effects of shear deformation, rotary inertia, Poison's ratio, extensional-bending-twisting coupling resulting from the anisotropy of the composite material.

Although most prior research has concentrated on the free vibration analysis of composite laminated beams, the dynamic analysis of these beams under dynamic forces has received limited attention. To the authors' knowledge, there are no existing studies that provide analytical closed-form solutions for the dynamic analysis of composite asymmetric laminated Timoshenko beams subjected to harmonic bending and twisting excitations. Therefore, the previous work of Hjaji and Nagiar (2023) is extended in this study to investigate the fully coupled of extensional-flexural-torsional-shear vibration responses by developing exact closed-form solutions for asymmetric laminated beams with rectangular cross-sections under various harmonic bending-twisting excitations. The four coupled governing equations and corresponding boundary conditions for these beams were derived using Hamilton’s variational principle. This study also examines the impact of shear deformation, rotary inertia, Poisson’s ratio, fiber orientation and extension-bending-shear-torsion coupling on natural frequencies, as well as quasi-static and steady-state dynamic responses. The exact closed-form solutions for axial and transverse deformations, bending and torsional rotations presented here are effective and suitable for analyzing the forced fully coupled responses of composite asymmetric laminated beams subjected to different harmonic excitations.

The significance of this research extends beyond theoretical analysis, encompassing a wide range of practical applications. The exact closed-from solutions developed for coupled extension-bending-shear-torsion dynamic responses of asymmetric laminated Timoshenko beams have significant implications across various engineering fields. For different beam types, this model can be extended to functionally graded materials (FGMs), which are important in structures such as turbine blades that experience variable material properties across their thickness. Furthermore, the methodology can be applied to hybrid laminated beams, which combine multiple materials, including metal and composites, to enhance structural performance in aerospace and automotive designs. Additionally, the framework is applicable to curved laminated beams, common in bridge arches and aircraft fuselage frames, where curvature introduces complexities in vibration behavior due to the interaction between extension, bending, and

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torsion. In terms of loading conditions, this model is adept at handling random dynamic loads, such as wind and seismic forces, which are crucial for assessing the integrity of civil infrastructure like bridges and offshore platforms. The solution is also applicable for analyzing moving loads, such as those imposed by vehicles or trains traversing bridges, which induce intricate vibration patterns. Moreover, the strength of the methodology makes it suitable for cases involving impact loading, an essential consideration in crashworthiness assessments and the design of protective structures. When considering nonlinear effects, the exact solution can be adapted for geometrically nonlinear behavior, particularly relevant for lightweight, flexible aerospace structures where significant nonlinear coupling may occur. The model can also accommodate nonlinear material behavior, such as viscoelastic or hyper elastic responses, allowing for the analysis of time-dependent effects in structural components. Incorporating nonlinear damping further enhances the model's utility, making it suitable for vibration control in civil and mechanical systems subjected to dynamic conditions.

Mathematical Formulation

A thin multi-layered composite beam ofspan L and rectangular cross section $(b \times h)$, where thickness h and width b, is considered as shown in Figure (1). The right-handed Cartesian coordinate system $(X, Y, Z)$is defined on the mid-plane of the composite beam, where X axis is taken along the beam axis, Y axis and Z axis is along the width and thickness of the composite beam, respectively.The mathematical model of the extension-bending-shear-torsion coupled asymmetric/antisymmetric composite laminated beam is based on the following assumptions:

- The present formulation is applicable to composite asymmetric laminated beams of rectangular cross-sections.
- The material of each layer exhibits linearly elastic behavior and has three planes of material symmetry.
- Each layer has a uniform thickness.
- Each layer is thin and all layers are perfectly bonded together.
- Displacements, strains and rotations are assumed small.

○ The beam cross-section remains planar but does not remain perpendicular to the centroidal axis after deformation, i.e., the transverse shear deformation of the mid-surface of the cross-section is incorporated in the assumed kinematics.
○ The effects due to material anisotropy and rotary inertia are taken into account.
○ Only the steady state dynamic response is required.
○ Damping is neglected in the formulation.

The axial, transverse displacements, bending and torsional rotationsfunctions for a general point$p(x,z)$ of height z from the centroidal axis of composite laminated beam based on first-order shear deformation theory ($FSDT$) take the following forms:

$$u_p(x,z,t) = u(x,t) + z\theta(x,t) \quad (1)$$

$$v_p(x,z,t) = z\phi(x,t) \quad (2)$$

$$w_p(x,z,t) = w(x,t) \quad (3)$$

Where $u(x,t)$, $v(x,t)$ and $w(x,t)$ are the axial, lateral and transverse displacements of a point $p(x,z)$on the mid-plane in the (X,Y,Z) directions, respectively, $u_p(x,z,t), v_p(x,z,t)$ and $w_p(x,z,t)$are the axial, lateral and transverse displacements of point p, respectively, $\theta(x,t)$ and $\phi(x,t)$are the rotations of the normal to the mid-plane about the $X$and$Y$ axes, respectively, and x and t are spanwise coordinate and time, respectively.

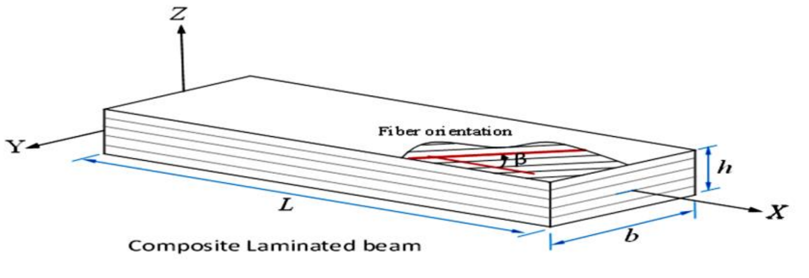

Figure 1. A composite laminated beam of rectangular cross-section

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The strain-displacement relations of the composite laminated beam associated with the small-displacement theory are given as:

$$\varepsilon_{xx}=\frac{\partial u_p(x,t)}{\partial x}=u'(x,t)+z\theta'(x,t)=\varepsilon_{xo}+zk_x \tag{4}$$

$$\gamma_{xz}=\frac{\partial w_p(x,t)}{\partial x}+\frac{\partial u_p(x,t)}{\partial z}=w'(x,t)+\theta(x,t) \tag{5}$$

$$\gamma_{xy}=\frac{\partial u_p(x,t)}{\partial y}+\frac{\partial v_p(x,t)}{\partial x}=z\phi'(x,t)=zk_{xy} \tag{6}$$

where$\varepsilon_{xo}=\partial u/\partial x$is the midplane axial strain, $k_x=\partial\theta/\partial x$is the midplane curvature, and $k_{xy}=\partial\phi/\partial x$ is the twisting curvature.
For asymmetric laminated beam, the constitutive equation is written in matrix form as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}_{6\times1} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}_{6\times6} \begin{Bmatrix} \varepsilon_{xo} \\ \varepsilon_{yo} \\ \gamma_{xy} \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix}_{6\times1} \tag{7}$$

where$N_x, N_y$and$N_{xy}$are the in-plane normal and shear forces, M_x, M_y and M_{xy} are the bending and twisting moments, $\varepsilon_{xo}, \varepsilon_{yo}$ and γ_{xy} are the mid-plane axial and shear strains, $k_x, k_y,$ and k_{xy} are the bending and twisting curvatures, respectively.A_{ij}, B_{ij} and D_{ij} denote the extensional, coupling and bending stiffnesses, respectively, which are functions of laminate ply orientation, stack sequence and material properties, are given as:

$$A_{ij},B_{ij},D_{ij}=\int_{-h/2}^{h/2}\left[\overline{Q}_{ij}\right](1,z,z^2)\,dz,\ \ (for\, i,j=1,2,6) \tag{8}$$

where the transformed reduced stiffness's $\overline{Q}_{ij}$and$\overline{Q}_{55}$are given by the following expressions (Jone 1975):

$$\overline{Q}_{11}=Q_{11}c^4+2(Q_{12}+2Q_{66})s^2c^2+Q_{22}s^4$$
$$\overline{Q}_{12}=(Q_{11}+Q_{22}-4Q_{66})s^2c^2+Q_{12}(s^4+c^4)$$
$$\overline{Q}_{22}=Q_{11}s^4+2(Q_{12}+2Q_{66})s^2c^2+Q_{22}s^2c^2$$
$$\overline{Q}_{16}=(Q_{11}-Q_{12}-2Q_{66})sc^3+(Q_{12}-Q_{22}+2Q_{66})s^3c$$

$$\overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})s^3c + (Q_{12} - Q_{22} + 2Q_{66})sc^3$$
$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4)$$

where$s = sin\beta,\ c = cos\beta,\ \beta$ is the angle between the fiber direction and longitudinal axis of the beam,and $Q_{11}, Q_{12}, Q_{22}$and $Q_{66}$are the stiffness constants and are given in terms of engineering elastic constants (Jun 2008)as:

$$Q_{11} = E_{11}/(1 - \upsilon_{12}\upsilon_{21}),\ Q_{12} = \upsilon_{12} E_{22}/(1 - \upsilon_{12}\upsilon_{21}) = \upsilon_{21} E_{11}/(1 - \upsilon_{12}\upsilon_{21}), Q_{22} = E_{22}/(1 - \upsilon_{12}\upsilon_{21})\text{ , and } Q_{66} = G_{12}.$$

in which the constants $E_{11}, E_{22}$are Elastic moduli, $G_{12}, G_{13}, G_{23}$are shear moduli, and $\upsilon_{12},\ \upsilon_{21}$are Poison ratios measured in the principal axes of the layer.
The present formulation is based on the first order shear deformation beam theory, the effect of transverse shear deformation is taken into account, then, the transverse shear force per unit length $Q_{xz}$is given (Jun 2008)by:

$$Q_{xz} = A_{55}\gamma_{xz} = A_{55}(w' + \theta) \qquad (9)$$

where$\overline{Q}_{55} = G_{13}c^2 + G_{23}s^2$, $A_{55} = k_s \int_{-h/2}^{h/2} \overline{Q}_{55} dz$, and$k_s$is the shear correction factor which taken as 5/6 to account for the parabolic variation of the transverse shear stresses.
When the laminated composite beam is subjected to axial force, bending and twisting moments. Then, the lateral in-plane forces and moment in Y direction are negligible and are set to zero, i.e., $N_y = N_{xy} = M_y = 0$. In order to account for Poisson's ratios, the mid-plane strains $\varepsilon_{yo}, \gamma_{xy}$ and curvaturesk_{yy},$k_{xy}$are assumed to be nonzero. For asymmetric/antisymmetric laminated beams, the extensional, bending and twisting responsesare fully coupled. Thus, equation (7) can be rewritten [6] as:

$$\begin{Bmatrix} N_x \\ M_x \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{A}_{11} & \overline{B}_{11} & \overline{B}_{16} \\ \overline{B}_{11} & \overline{D}_{11} & \overline{D}_{16} \\ \overline{B}_{16} & \overline{D}_{16} & \overline{D}_{66} \end{bmatrix} \begin{Bmatrix} u' \\ \theta' \\ \phi' \end{Bmatrix} \qquad (10)$$

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Where $\begin{bmatrix} \overline{A}_{11} & \overline{B}_{11} & \overline{B}_{16} \\ \overline{B}_{11} & \overline{D}_{11} & \overline{D}_{16} \\ \overline{B}_{16} & \overline{D}_{16} & \overline{D}_{66} \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & B_{16} \\ B_{11} & D_{11} & D_{16} \\ B_{16} & D_{16} & D_{66} \end{bmatrix} -$

$$\begin{bmatrix} A_{12} & A_{16} & B_{12} \\ B_{12} & B_{16} & D_{12} \\ B_{26} & B_{66} & D_{26} \end{bmatrix} \begin{bmatrix} A_{22} & A_{26} & B_{22} \\ A_{26} & A_{66} & B_{26} \\ B_{22} & B_{26} & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} A_{12} & A_{16} & B_{12} \\ B_{12} & B_{16} & D_{12} \\ B_{26} & B_{66} & D_{26} \end{bmatrix}^{T}.$$

If the influence of Poisson ratio is ignored, the stiffness coefficients $(\overline{A}_{11}, \overline{B}_{11}, \overline{B}_{16}, \overline{D}_{11}, \overline{D}_{16}, \overline{D}_{66})$ in equations (10) are then replaced by the stiffness coefficients $(A_{11}, B_{11}, B_{16,} D_{11}, D_{16}, D_{66})$, respectively.

Energy Expressions and Variational Formulation

The total kinetic energy of the composite laminated beam is given (Hjaji et. al 2017) as:

$$T = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2} \rho\left[\dot{u}_p^2 + \dot{v}_p^2 + \dot{w}_p^2\right] bdzdx$$

$$= \frac{1}{2}\int_0^L \rho\left[I_1\dot{u}^2 + 2I_2\dot{u}\dot{\theta} + I_3\dot{\theta}^2 + I_3\dot{\phi}^2 + I_1\dot{w}^2\right] b\, dx \quad (11)$$

in which the dot denotes the derivative with respect to time *t*, and the densities I_1, I_2 and I_3 of the composite laminated beam are introduced by substituting equations (1-3) into above equation as:

$$I_1, I_2, I_3 = \int_{-h/2}^{h/2} \rho[1, z, z^2]\, dz = \sum_{k=1}^{m} \rho_k \left[(z_k - z_{k-1}), \frac{1}{2}\left(z_k^2 - z_{k-1}^2\right), \frac{1}{3}\left(z_k^3 - z_{k-1}^3\right)\right] \quad (12)$$

where ρ_k is the mass density of the k^{th} layer.

The internal strain energy stored in the composite beam is given as:

$$U_s = \frac{1}{2}\int_0^L \int_A \left[\sigma_x \varepsilon_{xx} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz}\right] dAdx$$

$$= \frac{1}{2}\int_0^L \left[N_x \varepsilon_{xo} + M_x k_x + M_{xy} k_{xy} + Q_{xz}\gamma_{xz}\right] bdx$$

From equation (10) and above equation, gives:

$$U_s = \frac{1}{2}\int_0^L \left[\overline{A}_{11} {u'}^2 + A_{55} {w'}^2 + \overline{D}_{11} {\theta'}^2 + \overline{D}_{66} {\phi'}^2 + 2\overline{B}_{11} u'\theta' + 2\overline{B}_{16} u'\phi' + 2\overline{D}_{16} \theta'\phi' + 2A_{55}\omega'\theta + A_{55}\theta^2\right] bdx \quad (13)$$

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The potential energy of the applied forces is expressed as:

$$V_e = -\int_0^L \left[q_x(x,t)u(x,t) + q_z(x,t)w(x,t) + m_x(x,t)\theta(x,t) + m_{xy}(x,t)\phi(x,t)\right]dx - \left[P_x(x,t)u(x,t) + P_z(x,t)w(x,t) + M_x(x,t)\theta(x,t) + M_{xy}(x,t)\phi(x,t)\right]_0^L \quad (14)$$

where $q_x(x,t)$ and $q_z(x,t)$ are the distributed axial and transverse harmonic forces, $m_x(x,t)$, $m_{xy}(x,t)$ are the distributed harmonic bending and twisting moments, $P_x(x,t)$, $P_z(x,t)$ are the concentrated axial and transverse harmonic forces, $M_x(x,t), M_{xy}(x,t)$ are the concentrated harmonic bending and twisting moments applied at composite beam ends (i.e., $x = 0, L$).
The dynamic coupled equations governing the fully extension-bending-shear-torsion coupled responses of composite asymmetric laminated beam subjected to various harmonic forces and moments can be derived using the following Hamilton variational principle, which can be written as:

$$\int_{t_1}^{t_2} \delta(T - \Pi)\, dt = 0, \qquad \text{for } \delta u(x,t) = \delta w(x,t) = \delta\theta(x,t) = \delta\phi(x,t) = 0, \text{ at } t = t_1 \text{and} t_2 \quad (15)$$

where t_1 and$t_2$are two arbitrary time variables and δ denotes the first variation, and Π is the total potential energy which is the sum of the internal strain energy and the potential energy for the applied loads, i.e., $\Pi = U_s + V_s$.By substituting equations (11), (13) and (14) into Hamilton principle in (15), performing integration by parts, the governing coupled equations are then obtained as:

$$\overline{A}_{11}u''(x,t) - I_1\ddot{u}(x,t) + \overline{B}_{11}\theta''(x,t) - I_2\ddot{\theta}(x,t) + \overline{B}_{16}\phi''(x,t) = q_x(x,t)/b \quad (16)$$

$$A_{55}w''(x,t) - I_1\ddot{w}(x,t) + A_{55}\theta'(x,t) = q_z(x,t)/b \quad (17)$$

$$\overline{B}_{11}u''(x,t) - I_2\ddot{u}(x,t) - A_{55}w'(x,t) + \overline{D}_{11}\theta''(x,t) - A_{55}\theta(x,t) - I_3\ddot{\theta}(x,t) + \overline{D}_{16}\phi''(x,t) = m_x(x,t)/b \quad (18)$$

$$\overline{B}_{16}u(x,t) + \overline{D}_{16}\theta''(x,t) - I_3\ddot{\phi}(x,t) + \overline{D}_{66}\phi(x,t) = m_{xy}(x,t)/b \quad (19)$$

The related boundary conditions are:

$$\left[b\left(\overline{A}_{11}u' + \overline{B}_{11}\theta' + \overline{B}_{16}\phi'\right) + N_x(x,t)\right]_0^L \delta u(x)|_0^L = 0 \quad (20)$$

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$$[b(A_{55}w' + A_{55}\theta) + P_z(x,t)]_0^L \, \delta w(x)|_0^L = 0 \qquad (21)$$

$$\left[b\left(\overline{D}_{11}\theta' + \overline{B}_{11}u' + \overline{D}_{16}\phi'\right) + M_x(x,t)\right]_0^L \delta\theta(x)|_0^L = 0 \qquad (22)$$

$$\left[b\left(\overline{D}_{66}\phi' + \overline{B}_{16}u' + \overline{D}_{16}\theta'\right) + M_{xy}(x,t)\right]_0^L \delta\phi(x)|_0^L = 0 \qquad (23)$$

The set of governing coupled equations (16-19) and associated boundary conditions (20-23) describes the behavior of asymmetric laminated composite beam under various dynamic excitations. These coupled equations account for the complex interactions between axial, bending, and twisting deformations, influenced by the anisotropic material properties of laminated composites.

Expressions for Forces and Displacement Functions

The composite beam illustrated in Figure (2) is assumed subjected to general applied harmonic forces and moments within the beam:

$$\left[q_x(x,t), q_z(x,t), m_x(x,t), m_{xy}(x,t)\right] = \left[\overline{q}_x(x), \overline{q}_z(x), \overline{m}_x(x), \overline{m}_{xy}(x)\right]e^{i\Omega t} \qquad (24)$$

and the harmonic forces and moments applied at composite beam both ends ($x = 0, L$):

$$\left[P_x(x_e,t), P_z(x_e,t), M_x(x_e,t), M_{xy}(x_e,t)\right] = \left[\overline{P}_x(x_e), \overline{P}_z(x_e), \overline{M}_x(x_e), \overline{M}_{xy}(x_e)\right]e^{i\Omega t}, \ \text{for} x_e = 0, L \qquad (25)$$

where Ω is the circular exciting frequency of the applied harmonic forces and moments, $i = \sqrt{-1}$ is the imaginary constant.

Under the given harmonic forces and moments, the displacement and rotation fields corresponding to the steady state dynamic response are assumed to take the following form:

$$[u(x,t), w(x,t), \theta(x,t), \phi(x,t)] = \left[\overline{U}(x), \overline{W}(x), \overline{\theta}(x), \overline{\Phi}(x)\right]e^{i\Omega t} \qquad (26)$$

where $\overline{U}(x), \overline{W}(x), \overline{\theta}(x)and\overline{\Phi}(x)$ are the amplitude space functions for axial displacement, transverse displacement, bending and twisting rotations, respectively. Since the present formulation is designed to capture only the steady-state response of the coupled system of equations, the displacement fields proposed in the equation (26) disregard the transient component of the response.

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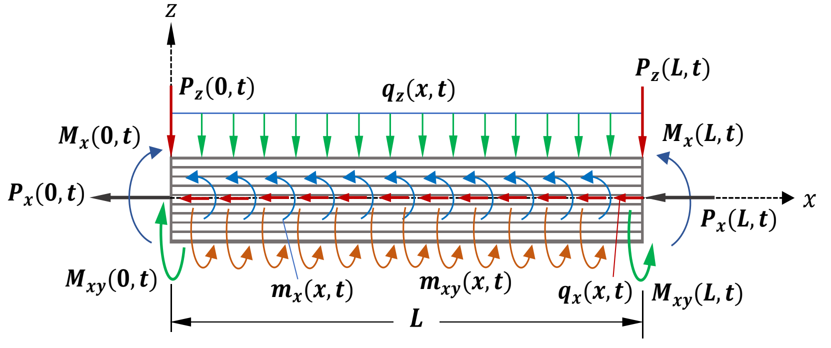

Figure 2. A composite under various harmonic forces and moments

Governing Field Equations

From equations (24-26) and by substituting into equations (16-19) and (20-23), the fully extension-bending-shear-torsion coupled equations of motion are obtained in matrix form as:

$$\begin{bmatrix} M_{11} & 0 & M_{13} & M_{14} \\ 0 & M_{22} & M_{23} & 0 \\ M_{13} & M_{23} & M_{33} & M_{34} \\ M_{14} & 0 & M_{34} & M_{44} \end{bmatrix} \begin{Bmatrix} \overline{U}(x) \\ \overline{W}(x) \\ \overline{\theta}(x) \\ \overline{\Phi}(x) \end{Bmatrix} = \begin{Bmatrix} \overline{q}_x(x)/b \\ -\overline{q}_x(x)/b \\ \overline{m}_x(x)/b \\ \overline{m}_{xy}(x)/b \end{Bmatrix} \qquad (27)$$

in which $M_{11} = (I_1\Omega^2 + \overline{A}_{11}\mathcal{D}^2)$, $M_{13} = (I_2\Omega^2 + \overline{B}_{11}\mathcal{D}^2)$, $M_{14} = \overline{B}_{16}\mathcal{D}^2$, $M_{22} = -(I_1\Omega^2 + A_{55}\mathcal{D}^2)$, $M_{23} = -A_{55}\mathcal{D}$, $M_{33} = (I_3\Omega^2 + \overline{D}_{11}\mathcal{D}^2 - A_{55})$, $M_{34} = \overline{D}_{16}\mathcal{D}^2$, $M_{44} = (I_3\Omega^2 + \overline{D}_{66}\mathcal{D}^2)$,where $\mathcal{D}$ is the differential operator, i.e.,$\mathcal{D} \equiv \frac{d}{dx}$, and $\mathcal{D}^2 \equiv \frac{d^2}{dx^2}$.

Equation (27) governs the extension-bending-shear-torsion coupled response for asymmetric laminated beams under various harmonic excitations. The present formulation focuses on developing the exact closed-form solutions for the steady state dynamic response governed by the four coupled equations provided in equation (27).

Closed-Form Solutions of Coupled Equations

The total closed-form solutions of the coupled field equations in (27) consist of two parts: the homogeneous solution and the particular solution.

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The homogeneous solution of the governing coupled field equations$\overline{U}_h(x)$, $\overline{W}_h(x)$, $\overline{\theta}_h(x)$and$\overline{\Phi}_h(x)$is obtained by setting the loading terms in the field equations to zero, i.e., $\overline{q}_x(x) = \overline{q}_z(x) = \overline{m}_x(x) = \overline{m}_{xy}(x) = 0$. The homogeneous solution of the displacement space functions is then assumed to take the following form:

$$\langle \overline{\chi}_h(x) \rangle_{1\times 4} = \langle c_1 \quad c_2 \quad c_3 \quad c_4 \rangle_{1\times 4}\, e^{m_i x} = \sum_{i=1}^{4} \langle C \rangle_{i,1\times 4}\; e^{m_i x} \tag{28}$$

where$\langle \overline{\chi}_h(x) \rangle_{1\times 4} = \langle \overline{U}_h(x) \overline{W}_h(x) \quad \overline{\theta}_h(x) \quad \overline{\Phi}_h(x) \rangle_{1\times 4}$, is the homogeneous solution of extensional, transverse displacements, bending and twisting rotations andthe vector of unknown constants is$\langle C \rangle_{i,1\times 4} = \langle c_1 \quad c_2 c_3 \quad c_4 \rangle_{i,4\times 1}$. From equation (28), by substituting into equations in (27), a non-trivial solution is obtained by setting the determinant of the matrix to zero, resulting in eighth-order polynomial equation of the form:

$$p_4 m_i^8 + p_3 m_i^6 + p_2 m_i^4 + p_1 m_i^2 + p_0 = 0 \tag{29}$$

where$p_0$through$p_4$are constants arising from the expansion of the determinant. These constants are obtained as:

$$p_o = \Omega^6 I_1 I_3 [A_{55} I_1 + \Omega^2 (I_2^2 - I_1 I_3)],$$

$$p_1 = \Omega^4 \big[\Omega^2 \big(\overline{D}_{66} I_1 I_2^3 + 2\overline{B}_{11} I_1 I_2 I_3 + A_{55} I_2^2 I_3 - \overline{D}_{11} I_1^2 I_3\big) - I_1 \big(\overline{D}_{66} I_1 + A_{55} I_3 + \overline{A}_{11} I_3\big)(I_3 \Omega^2 - A_{55}) - A_{55}^2 I_1 I_3\big],$$

$$p_2 = \Omega^2 \Big[\Omega^2 \big(A_{55} I_2 + \overline{B}_{11} I_1\big)\big(\overline{D}_{66} I_2 + \overline{B}_{11} I_3\big) - \big(\overline{A}_{11} \overline{D}_{66} I_1 + A_{55} \overline{D}_{66} I_1 + \overline{A}_{11} A_{55} I_3\big)(I_3 \Omega^2 - A_{55}) - I_1^2 \Omega^2 \left(\overline{D}_{11} \overline{D}_{66} - \overline{D}_{16}^2\right) - \overline{D}_{11} I_1 I_3 \Omega^2 \big(\overline{A}_{11} + A_{55}\big) - \overline{A}_{55}^2 \Omega^2 \big(\overline{D}_{66} I_1 + \overline{A}_{11} I_3\big) + I_2 \Omega^2 \big(\overline{B}_{11} \overline{D}_{66} I_1 - \overline{B}_{16} \overline{D}_{16} I_1 + A_{55} \overline{B}_{11} I_3\big) - \overline{B}_{16} I_1 \big(\overline{D}_{16} I_2 \Omega^2 - \overline{B}_{16} I_3 \Omega^2 + A_{55} \overline{B}_{16}\big)\Big],$$

$$p_3 = \Omega^2 \Big[\big(\overline{B}_{11} \overline{D}_{66} - \overline{B}_{16} \overline{D}_{16}\big)\big(A_{55} I_2 + \overline{B}_{11} I_1 + \overline{B}_{16} I_1\big) - I_1 \big(\overline{A}_{11} + A_{55}\big)\left(\overline{D}_{11} \overline{D}_{66} - \overline{D}_{16}^2\right) - \overline{A}_{11} A_{55} I_3 \big(\overline{D}_{11} + \overline{D}_{66}\big) + A_{55} \overline{B}_{11} \big(\overline{D}_{66} I_2 + \overline{B}_{11} I_3\big) - A_{55} \overline{B}_{16} \big(\overline{D}_{16} I_2 - \overline{B}_{16} I_3\big)\Big], \text{ and}$$

$$p_4 = A_{55} \left[\overline{A}_{11} \left(\overline{D}_{16}^2 - \overline{D}_{11} \overline{D}_{66}\right) + \overline{B}_{11} \big(\overline{B}_{11} \overline{D}_{66} - 2\overline{B}_{16} \overline{D}_{16}\big) + \overline{B}_{16}^2 \overline{D}_{11}\right].$$

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Equation (29) has eight nonzero distinct roots denoted as m_i (for $i = 1,2,3, \ldots,8$). For each root m_i, there is a corresponding set of constants $\langle C\rangle_{i,1\times4} = \langle c_1 \quad c_2 \quad c_3 \quad c_4\rangle_{i,1\times4}$. By substituting these values back into the homogeneous coupled system of equations in (27), the constants $c_{1,i}$, $c_{2,i}$ and $c_{4,i}$ can be expressed in terms of constants $c_{3,i}$ through the relations $c_{1,i} = G_{1,i}c_{3,i}$, $c_{2,i} = G_{2,i}c_{3,i}$, $c_{3,i} = G_{3,i}c_{3,i}$ and $c_{4,i} = G_{4,i}c_{3,i}$ (for $i = 1,2,3, \ldots,8$), respectively, where

$$G_{1,i} = -\left(\frac{m_i^2(\overline{B}_{11}+\overline{B}_{16}G_{4,i})+I_2\Omega^2}{(I_1\Omega^2+\overline{A}_{11}m_i^2)}\right), \quad G_{2,i} = \frac{-A_{55}m_i}{(A_{55}m_i^2+I_1\Omega^2)}, G_{3,i} = 1.0,$$

$$\text{and } G_{4,i} = \left(\frac{(I_2\Omega^2+\overline{B}_{11}m_i^2)^2-\lambda(I_1\Omega^2+\overline{A}_{11}m_i^2)}{\overline{D}_{16}m_i^2(I_1\Omega^2+\overline{A}_{11}m_i^2)-\overline{B}_{16}m_i^2(I_2\Omega^2+\overline{B}_{11}m_i^2)}\right)$$

in which $\lambda = \left(I_3\Omega^2 + \overline{D}_{11}m_i^2 - A_{55} - A_{55}m_iG_{2,i}\right)$.

The homogeneous solutions for the extensional, transverse displacements $\overline{U}_h(x)$, $\overline{W}_h(x)$, bending rotation $\overline{\theta}_h(x)$, and twisting angle $\overline{\phi}_h(x)$ given in equation (28) are expressed in matrix form as:

$$\{\overline{\chi}_h(x)\}_{4\times1} = [\overline{G}]_{4\times8}[\overline{E}(x)]_{8\times8}\{\overline{c}_{3,i}\}_{8\times1} \tag{30}$$

where $$[\overline{G}]_{4\times8} = \begin{bmatrix} \overline{G}_{1,1} & \overline{G}_{1,2} & \overline{G}_{1,3} & \ldots.. & \overline{G}_{1,8} \\ \overline{G}_{2,1} & \overline{G}_{2,2} & \overline{G}_{2,3} & \ldots.. & \overline{G}_{2,8} \\ \overline{G}_{3,1} & \overline{G}_{3,2} & \overline{G}_{3,3} & \ldots.. & \overline{G}_{3,8} \\ \overline{G}_{4,1} & \overline{G}_{4,2} & \overline{G}_{4,3} & \ldots.. & \overline{G}_{4,8} \end{bmatrix}_{4\times8},$$

$[\overline{E}(x)]_{8\times8}$ is a diagonal matrix consisting of exponential functions e^{m_ix} (for $i = 1,2,3,\ldots,8$), and the vector of unknown constants is $\langle\overline{c}_{3,i}\rangle_{1\times8} = \langle c_{3,1} \quad c_{3,2} \quad c_{3,3} \quad \ldots\ldots \quad c_{3,8}\rangle_{1\times8}$ is determined from the problem boundary conditions.

For a composite laminated beam subjected to uniform distributed harmonic excitations:

$$[\overline{q}_x(x), \overline{q}_z(x), \overline{m}_x(x), \overline{m}_{xy}(x)] = [\overline{q}_x, \overline{q}_z, \overline{m}_x, \overline{m}_{xy}]e^{m_ix} \tag{31}$$

The corresponding particular solution $\{\overline{\chi}_p\}_{4\times1}$ of the coupled field equations is obtained as:

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$$\left\{\overline{\chi}_p\right\}_{4\times1} = \left\{\begin{matrix} \left[(\Omega^2 I_3 - A_{55})\overline{q}_x - I_2\Omega^2\overline{m}_x\right]/[b\Omega^2(\Omega^2 I_1 I_3 - I_1 A_{55} - I_2^2\Omega^2)] \\ -\overline{q}_z/bI_1\Omega^2 \\ \left(\overline{m}_x I_1 - \overline{q}_x I_2\right)/[b\Omega^2(\Omega^2 I_1 I_3 - I_1 A_{55} - I_2^2\Omega^2)] \\ \overline{m}_{xy}/bI_3\Omega^2 \end{matrix}\right\}_{4\times1} \tag{32}$$

where $\langle\overline{\chi}_p\rangle_{1\times4} = \langle\overline{U}_p\overline{W}_p \quad \overline{\theta}_p \quad \overline{\Phi}_p\rangle_{1\times4}$. Then, the total steady state dynamic response for the coupled field equations is determined by adding the homogeneous solution in equation (30) to the particular solution in equation (32), yielding:

$$\{\overline{\chi}(x)\}_{4\times1} = \left[\overline{G}\right]_{4\times8}\left[\overline{E}(x)\right]_{8\times8}\left\{\overline{c}_{3,i}\right\}_{8\times1} + \left\{\overline{\chi}_p\right\}_{4\times1} \tag{33}$$

where $\langle\overline{\chi}(x)\rangle_{1\times4} = \langle\overline{U}(x)\overline{W}(x) \quad \overline{\theta}(x) \quad \overline{\Phi}(x)\rangle_{1\times4}$, and the unknown constants$\left\{\overline{c}_{3,i}\right\}_{1\times8}$ appearing in equation (33) are obtained by applying the beam boundary conditions.

The boundary conditions for the asymmetric laminated beam considered in the present formulation are given in Table (1). By using the appropriate boundary conditions for the composite beam, the complete closed-form solutions for clamped-free (*CF*), simply-supported (*SS*), clamped-clamped (*CC*) and clamped-pinned (*CP*) asymmetric laminated beams under various harmonic excitations are conducted in Table (1).

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Table (1): The exact closed-from solutions for asymmetric laminated beams for different boundary conditions

Beam type	***Boundary conditions***
CF	$\boldsymbol{\overline{U}(0) = \overline{W}(0) = \overline{\theta}(0) = \overline{\Phi}(0) = 0}$, $\boldsymbol{\left[\overline{A}_{11}\overline{U'}(L) + \overline{B}_{11}\overline{\theta'}(L) + \overline{B}_{16}\overline{\Phi'}(L)\right] = -\overline{P}_x(L)/b}$ $A_{55}\left(\overline{W'}(L) + \overline{\theta}(L)\right) = -\overline{P}_z(L)/b$, $\left[\overline{B}_{11}\overline{U'}(L) + \overline{D}_{11}\overline{\theta'}(L) + \overline{D}_{16}\overline{\Phi'}(L)\right] = -\overline{M}_x(L)/b$ $\boldsymbol{b\left[\overline{B}_{16}\overline{U'}(L) + \overline{D}_{16}\overline{\theta'}(L) + \overline{D}_{66}\overline{\Phi'}(L)\right] = -\overline{M}_{xy}(L)/b}$
	Closed-form solution
	$\{\overline{\chi}_c(x)\}_{4x1}$=$\left[\overline{G}\right]_{4x8}\left[\overline{E}(x)\right]_{8x8}[\Psi_c]^{-1}_{8\times8}\{\overline{Q}_c\}_{8\times1} + \{\overline{\chi}_p\}_{4x1}$ where $[\Psi_c]^T_{8\times8} = \left[\{\overline{G}_{1,i}\}\vert\{\overline{G}_{2,i}\}\vert\{1\}\vert\{\overline{G}_{4,i}\}\vert\{\eta_{1,i}\}\vert\{\eta_{2,i}\}\vert\{\eta_{3,i}\}\vert\{\eta_{4,i}\}\right]^T_{8\times8}$, $\eta_{1,i} = m_i e^{m_i L}\left(\overline{A}_{11}\overline{G}_{1,i} + \overline{B}_{11} + \overline{B}_{16}\overline{G}_{4,i}\right)$, $\eta_{2,i} = e^{m_i L}\left(m_i\overline{G}_{2,i} + 1\right)$ $\eta_{3,i} = m_i e^{m_i L}\left(\overline{B}_{11}\overline{G}_{1,i} + \overline{D}_{11} + \overline{D}_{16}\overline{G}_{4,i}\right)$, $\eta_{4,i} = m_i e^{m_i L}\left(\overline{B}_{16}\overline{G}_{1,i} + \overline{D}_{16} + \overline{D}_{66}\overline{G}_{4,i}\right)$ $\langle\overline{Q}_c\rangle_{1\times8} = \left\langle -\overline{U}_P \middle\vert -\overline{W}_P \middle\vert -\overline{\theta}_P \middle\vert -\overline{\Phi}_P \middle\vert \frac{-\overline{P}_x(L)}{b} \middle\vert \frac{-\overline{P}_z(L)}{A_{55}b} - \overline{\theta}_P \middle\vert \frac{-\overline{M}_x(L)}{b} \middle\vert \frac{-\overline{M}_{xy}(L)}{b} \right\rangle_{1\times8}$
SS	***Boundary conditions***
	$\overline{U}(0) = \overline{W}(0) = \overline{\Phi}(0) = 0$, $\left[\overline{D}_{11}\overline{\theta'}(0) + \overline{B}_{11}\overline{U'}(0) + \overline{D}_{16}\overline{\Phi'}(0)\right] =$ $\overline{M}_x(0)/b$ $\left[\overline{A}_{11}\overline{U'}(L) + \overline{B}_{11}\overline{\theta'}(L) + \overline{B}_{16}\overline{\Phi'}(L)\right] = -\overline{P}_x(L)/b$, $\left[\overline{D}_{11}\overline{\theta'}(L) + \overline{B}_{11}\overline{U'}(L) + \overline{D}_{16}\overline{\Phi'}(L)\right] = -\overline{M}_x(L)/b$
	Closed-form solution
	$\{\bar{\chi}_{ss}(x)\}_{4x1}$=$\left[\overline{G}\right]_{4x8}\left[\overline{E}(x)\right]_{8x8}\left[\overline{\Psi}_{ss}\right]^{-1}_{8\times8}\langle\overline{Q}_{ss}\rangle_{1\times8} + \{\bar{\chi}_p\}_{4x1}$, where $[\overline{\Psi}_{ss}]^T_{8\times8} = \begin{bmatrix} \{\overline{G}_{1,i}\}_{8\times1}\{\overline{G}_{2,i}\}_{8\times1}\{\mu_{1,i}\}_{8\times1}\{\overline{G}_{4,i}\}_{8\times1}\{\mu_{2,i}\}_{8\times1}\{\mu_{3,i}\}_{8\times1} \\ \{\mu_{4,i}\}_{8\times1}\{\mu_{5,i}\}_{8\times1} \end{bmatrix}^T_{8\times8}$ $\mu_{1,i} = m_i\left(\overline{B}_{11}\overline{G}_{1,i} + \overline{D}_{11} + \overline{D}_{16}\overline{G}_{4,i}\right)$, $\mu_{2,i} = \eta_{1,i}$, $\mu_{3,i} = \overline{G}_{2,i}e^{m_i L}$, $\mu_{4,i} = \eta_{3,i}$ and$\mu_{5,i} = \overline{G}_{4,i}e^{m_i L}$, $\langle\overline{Q}_{ss}\rangle_{1\times8} = \left\langle -\overline{U}_p \middle\vert -\overline{W}_p \middle\vert \frac{\overline{M}_x(0)}{b} \middle\vert -\overline{\Phi}_p \middle\vert \frac{-\overline{N}_x(L)}{b} \middle\vert -\overline{W}_p \middle\vert \frac{-\overline{M}_x(L)}{b} \middle\vert -\overline{\Phi}_p \right\rangle_{1\times8}$

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Beam type	***Boundary conditions***
CC	$\overline{U}(0)=\overline{W}(0)=\overline{\theta}(0)=\overline{\Phi}(0)=0\,,\left[\overline{A}_{11}\overline{U'}(L)+\overline{B}_{11}\overline{\theta'}(L)+\bar{B}_{16}\overline{\Phi'}(L)\right]=\overline{P}_x(L)/b$ $\overline{W}(L)=\overline{\Phi}(L)=0\,,\left[\overline{D}_{11}\overline{\theta'}(L)+\overline{B}_{11}\overline{U'}(L)+\overline{D}_{16}\overline{\Phi'}(L)\right]=-\overline{M}_x(L)/b$
	Closed-form solution
	$\{\overline{\chi}_{cc}(x)\}_{4x1}=[\overline{G}]_{4x8}[\overline{E}(x)]_{8x8}[\overline{\Psi}_{cc}]_{8\times8}^{-1}\langle\overline{Q}_{cc}\rangle_{1\times8}+\{\overline{\chi}_p\}_{4x1}$ where $[\overline{\Psi}_{cc}]_{8\times8}^{T}=\begin{bmatrix}\{\overline{G}_{1,i}\}_{8\times1}\{\overline{G}_{2,i}\}_{8\times1}\{1\}_{8\times1}\{\overline{G}_{4,i}\}_{8\times1}\{\overline{G}_{1,i}e^{m_iL}\}_{8\times1}\\ \{\mu_{3,i}\}_{8\times1}\{e^{m_iL}\}_{8\times1}\{\mu_{5,i}\}_{8\times1}\end{bmatrix}_{8\times8}^{T}\,,\langle\overline{Q}_{cc}\rangle_{1\times8}=\langle-\overline{U}_p\vert-\overline{W}_p\vert-\overline{\theta}_p\vert-\overline{\Phi}_p\vert-\overline{U}_p\vert-\overline{W}_p\vert-\overline{\theta}_p\vert-\overline{\Phi}_p\rangle_{1\times8}$
CP	***Boundary conditions***
	$\boldsymbol{\overline{U}(0)=\overline{W}(0)=\overline{\theta}(0)=\overline{\phi}(0)=0\,,\ \overline{U}(L)=\overline{W}(L)=\overline{\theta}(L)=\overline{\phi}(L)=0}$
	Closed-form solution
	$\{\bar{\chi}_{cp}(x)\}_{4x1}=[\overline{G}]_{4x8}[\overline{E}(x)]_{8x8}[\overline{\Psi}_{cp}]_{8\times8}^{-1}\langle\overline{Q}_{cp}\rangle_{1\times8}+\{\overline{\chi}_p\}_{4x1}$, where $[\overline{\Psi}_{cp}]_{8\times8}^{T}=\begin{bmatrix}\{\overline{G}_{1,i}\}_{8\times1}\{\overline{G}_{2,i}\}_{8\times1}\{1\}_{8\times1}\{\overline{G}_{4,i}\}_{8\times1}\{\eta_{1,i}\}_{8\times1}\{\mu_{3,i}\}_{8\times1}\\ \{\eta_{3,i}\}_{8\times1}\{\mu_{5,i}\}_{8\times1}\end{bmatrix}_{8\times8}^{T}$, and $\boldsymbol{\langle\overline{Q}_{cp}\rangle_{1\times8}=\left\langle-\overline{U}_p\middle\vert-\overline{W}_p\middle\vert-\overline{\theta}_p\middle\vert-\overline{\Phi}_p\middle\vert\frac{\overline{P}_x(L)}{b}\middle\vert-\overline{W}_p\middle\vert\frac{-\overline{M}_x(L)}{b}\middle\vert-\overline{\Phi}_p\right\rangle_{1\times8}}$

The exact solutions presented in Table (1) for different boundary conditions, including clamped-free, simply-supported, clamped-clamped, and clamped-pinned beams, provide comprehensive analytical expressions for the fully coupled extension-bending-shear-torsion vibration responses of asymmetric laminated beams. These solutions account for the effects of transverse shear deformation, rotary inertia, and material anisotropy, offering a versatile framework for analyzing dynamic behavior under various harmonic excitations. By applying these solutions, the dynamic responses of laminated beams can be accurately evaluated for different configurations without the need for numerical methods. This analytical approach is highly efficient and valuable for engineers dealing with complex beam structures in practical applications.

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Summary and Conclusion

- ✓ This paper presents exact closed-form solutions for the vibration analysis of asymmetric laminated Timoshenko beams under harmonic bending and twisting excitations.
- ✓ Using Hamilton's principle, four governing coupled differential equations are derived, incorporating first-order shear deformation theory, which accounts for transverse shear deformation, rotary inertia, Poisson's ratio, and material anisotropy.
- ✓ The formulated equations describe the coupled extension-bending-shear-torsion responses and are applicable to beams with various boundary conditions, including clamped-free, clamped-clamped, simply-supported, and clamped-pinned configurations.
- ✓ The derived closed-form solutions provide a robust analytical method for studying the fully coupled dynamic responses of laminated composite beams without requiring numerical methods such as finite element analysis.
- ✓ The formulation demonstrates the effects of material anisotropy, transverse shear deformation, and rotary inertia on beam dynamics, offering an efficient and accurate analytical tool for dynamic analysis in engineering applications.
- ✓ This study lays the groundwork for further exploration of more complex loading scenarios, non-linear behavior, and other beam configurations, making the method adaptable to a wide range of structural analysis problems.

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